REAL SINGULAR DEL PEZZO SURFACES AND THREEFOLDS FIBRED BY RATIONAL CURVES, I.

FABRIZIO CATANESE
UNIVERSITÄT BAYREUTH
FRÉDÉRIC MANGOLTE
UNIVERSITÉ DE SAVOIE

Abstract. Let $W \to X$ be a real smooth projective threefold fibred by rational curves. Kollár proved that if $W(\mathbb{R})$ is orientable a connected component N of $W(\mathbb{R})$ is essentially either a Seifert fibred manifold or a connected sum of lens spaces. Let $k := k(N)$ be the integer defined as follows: If $g\colon N \to F$ is a Seifert fibration, one defines $k := k(N)$ as the number of multiple fibres of g, while, if N is a connected sum of lens spaces, k is defined as the number of lens spaces different from $\mathbb{P}^3(\mathbb{R})$. Our Main Theorem says: If X is a geometrically rational surface, then $k \leq 4$. Moreover we show that if F is diffeomorphic to $S^1 \times S^1$, then $W(\mathbb{R})$ is connected and $k = 0$.

These results answer in the affirmative two questions of Kollár who proved in 1999 that $k \leq 6$ and suggested that 4 would be the sharp bound. We derive the Theorem from a careful study of real singular Del Pezzo surfaces with only Du Val singularities.

MSC 2000: 14P25, 14M20, 14J26.

Keywords: Del Pezzo surface, rationally connected algebraic variety, Seifert manifold, Du Val surface

Contents

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INTRODUCTION

In complex algebraic geometry there is an established principle that the Kodaira dimension of a smooth complex projective variety W of dimension n strongly influences the topology of the set $W(\mathbb{C})$ of its complex points. This principle is clearly manifest already in dimension 1, and related to other points of view, as the uniformization theorem, and the concept of curvature. This principle, although in a more difficult and complicated way, still goes on to hold in higher dimensions.

Let us assume now that W is a smooth real projective variety and let us consider the topology of the set $W(\mathbb{R})$ of its real points. In dimension 1, the connected components are just diffeomorphic to the circle S^1, and their number is not dictated by the genus (there is only the Harnack inequality which gives $g+1$ as upper bound for the number of connected components of $W(\mathbb{R})$).

So, there had been for some time the belief that the Kodaira dimension of W would not affect at all the topology of a connected component N of $W(\mathbb{R})$. This belief is contradicted already by the example of real algebraic surfaces of nonpositive Kodaira dimension (see for instance [Co14], [Sil89], [DIK00] and [Ko01]).

In a very interesting series of papers ([Ko98, Ko99a, Kol99b, Ko00]) János Kollár used the recent progress on the minimal model program for threefolds in order to understand the topology of the connected components $N \subset W(\mathbb{R})$, especially in the case where W has Kodaira dimension $-\infty$.

Our note takes the origin from some questions that Kollár set in the third article of the series ([Kol99b]), and we prove some optimal estimates that Kollár conjectured to hold.

The present note is mainly devoted to the proof of the following

Theorem 0.1. *Let X be a projective surface defined over $\mathbb{R}$. Suppose that X is geometrically rational with Du Val singularities. Then a connected component M of the topological normalization $\overline{X(\mathbb{R})}$ contains at most 4 Du Val singular points which are either not of type A^- or of type A^- but globally separating.*

Applying this result to rational curve fibrations over rational surfaces, we obtain the answer to two of the three questions set by Kollár (remark 1.2 of [Kol99b]). In a second note, with slightly different methods, we plan to answer also the third question.

Let us now explain these applications in more detail.

Let $f\colon W \to X$ be a real smooth projective threefold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable. Then, by [Kol99b, Theorem 1.1], a connected component $N \subset W(\mathbb{R})$ is a Seifert fibred manifold, or a connected sum of lens spaces, or obtained from one of the above by taking connected sums with a finite number of copies of $\mathbb{P}^3(\mathbb{R})$ and a finite number of copies of $S^1 \times S^2$. Note that in [HM05] and [HM05b] it was shown that all the manifolds N as above do indeed occur.

Note also that the connected sum $N_1 \# N_2$ is taken in the category of oriented manifolds, where in general $N_1 \# N_2$ is not homeomorphic to $N_1 \# -N_2$. But

for the particular choice $N_2 = \mathbb{P}^3(\mathbb{R})$ or $N_2 = S^1 \times S^2$, the connected sums $N_1 \# N_2$ and $N_1 \# -N_2$ are diffeomorphic, see e.g. [Hem76].

Take a decomposition $N = N' \#^a \mathbb{P}^3(\mathbb{R}) \#^b (S^1 \times S^2)$ with $a+b$ maximal and observe that this decomposition is unique by a theorem of Milnor [Mil62].

We shall focus our attention on the integer $k := k(N)$ defined as follows:

(1) if $g\colon N' \to F$ is a Seifert fibration, k denotes the number of multiple fibres of g;
(2) if N' is a connected sum of lens spaces, k denotes the number of lens spaces.

Observe that when N' is a connected sum of lens spaces, the number k is well defined (again by Milnor's theorem).

We can then apply the result of Theorem 0.1 concerning singular rational surfaces in order to answer one question of Kollár, [Kol99b, Remark 1.2 (1)].

Corollary 0.2. *Let $W \to X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X. Suppose that $W(\mathbb{R})$ is orientable. Then for each connected component $N \subset W(\mathbb{R})$, $k(N) \leq 4$.*

Note that Kollár showed in [Kol99b] the optimality of the above estimate in case 1).

The following theorem answers another question of Kollár, [Kol99b, Remark 1.2 (3)]

Theorem 0.3. *Let W be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X. Suppose that the fibration is defined over $\mathbb{R}$ and that $W(\mathbb{R})$ is orientable. Let $N \subset W(\mathbb{R})$ be a connected component which admits a Seifert fibration $g\colon N \to S^1 \times S^1$. Then g has no multiple fibres. Furthermore, X is then rational over $\mathbb{R}$ and $W(\mathbb{R})$ is connected.*

Section 1 is devoted to recalling the basic notions which come into play, especially the local and global separation properties of Du Val singularities of types A_μ. Hence the basic invariants m_i of a Du Val surface are defined.

Section 2 is the heart of the paper and contains a detailed description of the topological normalization of a real Du Val Del Pezzo surface with more than 4 singular points. The description is based on the classical representation of the quadric cone on the plane which transforms the hyperplane sections of the cone to parabolae whose axis has a given direction.

Section 3 proves Corollary 0.2, while Section 4 is devoted to the proof of Theorem 0.3.

1. Real Du Val surfaces

The aim of this section is to reduce the proof of Theorem 0.1 to the study of a certain kind of rational surfaces. The first part is close to the treatment in [Kol99b, Section 9].

On a surface, a rational double point is called a Du Val singularity. Over $\mathbb{C}$, these singularities are classified by their Dynkin diagrams, namely A_μ, $\mu \geq 1$, D_μ, $\mu \geq 4$, E_6, E_7, E_8.

Over $\mathbb{R}$, there are more possibilities. In particular, a surface singularity will be said to be *of type* A_μ^+ if it is real analytically equivalent to

$$x^2 + y^2 - z^{\mu+1} = 0, \ \mu \geq 1 \ ;$$

and *of type* A_μ^- if it is real analytically equivalent to

$$x^2 - y^2 - z^{\mu+1} = 0, \ \mu \geq 1 \ .$$

The type A_1^+ is real analytically isomorphic to A_1^-; otherwise, singularities with different names are not isomorphic. For μ odd, there is another real form of A_μ given by $x^2 + y^2 + z^{\mu+1} = 0$. We exclude this type of singular point because an isolated real point gives rise to $\emptyset$ on the minimal resolution.

Definition 1.1. *Let X be a projective surface. The surface X is called a* Du Val *surface if X has only rational double points as singularities.*

We want to use a suitable minimal model for X. In the minimal model program for real Du Val surfaces, the most useful statement for our purpose is the following description of the extremal contractions.

Theorem 1.2. [Kol99b, Th. 9.6] *Let X be a real Du Val surface, $\overline{NE}(X)$ be its cone of curves, and $R \subset \overline{NE}(X)$ be a K_X-negative extremal ray. Then R can be contracted. Furthermore, if $c \colon X \to Y$ is the contraction, c is one of the following:*

- *Y is a Du Val surface, c is birational, and $\rho(Y) = \rho(X) - 1$,*
- *Y is a smooth curve, $\rho(X) = 2$, and $c \colon X \to Y$ is a conic bundle,*
- *Y is a point, $\rho(X) = 1$ and X is a Du Val Del Pezzo surface (i.e. $-K_X$ is ample).*

To apply the minimal model program for our purposes, we need to understand the behavior of c when c is a birational contraction. We begin with a typical example.

Example 1.3. *Let Y be a real Du Val surface, $x \in Y$ be a smooth real point, and $\mu > 0$ be an integer. Blow-up Y at x, and denote by E_0 the exceptional curve of the blow-up $Y_0 \to Y$. Then take repeatedly the blow-up $Y_{l+1} \to Y_l$ at a general point on the exceptional curve E_l for $l = 0, 1, \ldots, \mu - 1$. The exceptional divisor of the composition of blow-ups $Y_\mu \to Y$ is a chain of rational curves whose configuration is of the form:*

Contracting the (-2)-curves $E_{\mu-l}$, $l = 0, 1, \ldots, \mu - 1$, we get a surface X with a singularity of type A_μ^-.

The interesting fact is that the birational contractions of Theorem 1.2 involve only this kind of construction (see [Kol99b, Th. 9.6]).

As we shall see in Section 3, bounding the number of certain singularities on $X(\mathbb{R})$ yields a bigger upper bound for $k(N)$ than the one stated in Corollary 0.2. In order to obtain this finer estimate we have to bound this number on each component of the topological normalization $\overline{X(\mathbb{R})}$ of $X(\mathbb{R})$, which we are going now to define.

Definition 1.4. *Let V be a simplicial complex with only a finite number of points $x \in V$ where V is not a manifold. Define the* topological normalization

$$\overline{n} \colon \overline{V} \to V$$

as the unique proper continuous map such that $\overline{n}$ is a homeomorphism over the set of points where V is a manifold and $\overline{n}^{-1}(x)$ is in one-to-one correspondence with the connected components of a good punctured neighborhood of x in V otherwise.

Observe that if V is pure of dimension 2, then $\overline{V}$ is a topological manifold. Indeed each point of $\overline{V}$ has a neighbourhood which is a cone over S^1.

Definition 1.5. *Let X be a real algebraic surface with isolated singularities, and let $x \in X(\mathbb{R})$ be a singular point of type $A_\mu^\pm$ with μ odd. The topological normalization $\overline{X(\mathbb{R})}$ has two connected components locally near x. We will say that x is* globally separating *if these two local components are on different connected components of $\overline{X(\mathbb{R})}$ and* globally nonseparating *otherwise.*

One can produce an arbitrarily high number of singular points of type A_μ^- by the construction of Example 1.3, but these singular points are globally nonseparating. Indeed, when μ is even, the singular point is in fact locally nonseparating, and when μ is odd, then the inverse image of the last $S^1 = E_\mu(\mathbb{R})$ yields a segment in $\overline{X(\mathbb{R})}$ connecting the two points. The key point for the sequel is the next lemma.

Definition 1.6. *Let X be a real Du Val surface, let*

$$(1) \quad \mathcal{P}_X := \operatorname{Sing} X \setminus \left\{x \textit{ of type } A_\mu^-,\ \mu \textit{ even}\right\} \\ \setminus \left\{x \textit{ of type } A_\mu^-,\ \mu \textit{ odd and } x \textit{ is globally nonseparating}\right\} .$$

We have

Lemma 1.7. [Kol99b, Cor. 9.7] *Let X be a real Du Val surface, let $\overline{n} \colon \overline{X(\mathbb{R})} \to X(\mathbb{R})$ be the topological normalization, and let $M_1, M_2, \dots, M_r$ be the connected components of $\overline{X(\mathbb{R})}$. The unordered sequence of numbers $m_i := \#(\overline{n}^{-1}(\mathcal{P}_X) \cap M_i)$, $i = 1, 2, \dots, r$ is an invariant of extremal birational contractions of Du Val surfaces.*

By Theorem 1.2 and Lemma 1.7, it suffices to prove Theorem 0.1 in the case when X is a conic bundle or a Del Pezzo surface with $\rho(X) = 1$. Conic bundles were analysed in [Kol99b, Section 9]. The remaining case is when X is a Del Pezzo surface. We still slightly reduce the problem to the case where X is a degree 1 Del Pezzo surface.

Lemma 1.8. *Let X be a real Du Val Del Pezzo surface possessing a smooth real point and having $\rho(X) = 1$. Then there exists a blow-up of X in smooth points yielding Z which is a conic bundle if $\deg X \geq 3$. Else we get Z a singular Del Pezzo surface of degree 1 with $\rho(Z) \leq 2$.*

Proof. Set $d := \deg X$. If $d \geq 3$, blow-up $(d-3)$ smooth points until you get a real cubic surface Z. The surface Z contains a real rational line L. We get $L \subset Z \subset \mathbb{P}^3$, and $\pi_L \colon \mathbb{P}^3 - L \to \mathbb{P}^2$ is a morphism and yields a real conic bundle. If $d = 2$ blow-up a smooth real point : $\rho(X)$ increases by 1. □

2. Singular Del Pezzo surfaces of degree 1

Recall that a Del Pezzo surface X is by definition a surface whose anticanonical divisor is ample. We add the adjective *Du Val* to emphasize that we allow X to have Du Val singularities (observe that for a Du Val surface, the canonical divisor is a Cartier divisor). Let X be a real Du Val Del Pezzo surface and let $p\colon S \to X$ be the minimal resolution of singularities. The smooth surface S has nef anticanonical divisor $-K_S = p^*(-K_X)$, and is called a *weak Del Pezzo surface* by many authors. As we saw in Section 1, we can assume the Del Pezzo surface X to have degree 1 by blowing up a finite number of pairs of conjugate imaginary smooth points and some real smooth point (there are several choices to do this), see Lemma 1.8. The anticanonical model of a Del Pezzo surface X of degree 1 is a ramified double covering $q\colon X \to Q$ of a quadric cone $Q \subset \mathbb{P}^3$ whose branch locus is the union of the vertex of the cone and a cubic section B not passing through the vertex, see e.g. [Dem80, Exposé V].

Remark that the pull-back by q of the vertex of the cone is a smooth point of X and let X' be the singular elliptic surface obtained from X by blowing up this smooth point. We denote by $\overline{n}\colon \overline{X'(\mathbb{R})} \to X'(\mathbb{R})$ the topological normalization of the real part.

We shall now make a series of considerations which will later lead to a proof of the following.

Proposition 2.1. *For each connected component $M \subset \overline{X'(\mathbb{R})}$,*

$$\#(\overline{n}^{-1}(\mathcal{P}_{X'}) \cap M) \leq 4\ .$$

Recall that Hirzebruch surfaces are the $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. The surface X' is a ramified double covering of the Hirzebruch surface $\mathbb{F}_2$ whose branch curve is the union of the unique section of negative selfintersection, the section at infinity Σ_∞, and a trisection B of the ruling $\mathbb{F}_2 \to \mathbb{P}^1$ which is disjoint from Σ_∞.

The cone Q is the weighted projective plane $\mathbb{P}(1,1,2)$ with coordinates (x_0, x_1, y_2), and X is the hypersurface in $\mathbb{P}(1,1,2,3)$ with coordinates (x_0, x_1, y_2, z) defined by

$$z^2 = y_2^3 + p_4(x_0, x_1)y_2 + q_6(x_0, x_1)\ .$$

We want to explain here the plane model of Q, in which the hyperplane sections of Q embedded in $\mathbb{P}^3$ by $H^0(\mathcal{O}_Q(2))$ correspond to parabolae tangent to the line at infinity $L_\infty = \{w = 0\}$ at the point $O := \{w = x = 0\}$ of the projective plane with coordinates (x, y, w). In other words, blow-up O and then the infinitely near point O' to O corresponding to the tangent of the line at infinity L_∞, and denote by $\tilde{Q}$ the resulting surface. Denote by E, E' the respective total transforms of O, O', and observe that $E = E' + E''$, E'' being a (-2)-curve. The linear system $H^0(\mathcal{O}_{\tilde{Q}}(2H - E - E'))$ maps $\tilde{Q}$ birationally onto the quadric cone $Q \subset \mathbb{P}^3$, contracting the proper transform $\tilde{L}_\infty$ of L_∞ and E'' to points. Since $\tilde{L}_\infty$ and E'' do not meet, first contracting $\tilde{L}_\infty$ yields the Hirzebruch surface $\mathbb{F}_2$, whose (-2)-section Σ_∞ is the image of the

curve E''. Let us write everything using the coordinates (x, y, w) in $\mathbb{P}^2$: then $H^0(\mathcal{O}_Q(1))$ corresponds to $H^0(\mathcal{O}_{\tilde{Q}}(H-E))$ spanned by w, x, whereas $y_2 =: yw$ completes w^2, wx, x^2 to a basis of $H^0(\mathcal{O}_{\tilde{Q}}(2H - E - E')) \cong H^0(\mathcal{O}_Q(2))$. Thus the morphism of $\tilde{Q}$ to $\mathbb{P}(1, 1, 2)$ is given by $x_0 := w$, $x_1 := x$, $y_2 := yw$.

The elliptic surface X' is the double cover of $\mathbb{F}_2$ branched on Σ_∞ and on the curve B corresponding to the curve of Q of equation $y^3 + p_4(x_0, x_1)y + q_6(x_0, x_1) = 0$. Thus the curve B corresponds to the plane curve $w^3y^3 + p_4(w, x)yw + q_6(w, x) = 0$ whose affine part has equation:

$$y^3 + p_4(1, x)y + q_6(1, x) = 0 \ . \tag{2}$$

Note that any parabola as above, i.e., a curve $C \in (2H - E - E')$ is disjoint from E'' (mapping to the vertex of the cone) unless it splits into two lines through the point O. In particular, we may always change coordinates in the affine plane so that C is transformed into the line $y = 0$. In order to understand with coordinates the geometry at infinity of parabolae as above, let us observe that $\mathbb{F}_2$ has two open sets isomorphic to $\mathbb{C} \times \mathbb{P}^1$. They have respective coordinates $\frac{x}{w} \in \mathbb{C}$, $(w, y) \in \mathbb{P}^1$, while on the other chart we have $\frac{w}{x} \in \mathbb{C}$, and homogeneous coordinates $(\frac{x^2}{w}, y)$ (in fact $\frac{x^2}{w}/w = (\frac{x}{w})^2$). The section Σ_∞ at infinity corresponds to the curve $E'' \subset \tilde{X}$ and is defined by $w = 0$ and $\frac{x^2}{w} = 0$ on the respective charts. Then a parabola $yw = a_0w^2 + a_1xw + a_2x^2$ is given by the equation

$$\frac{1}{\eta} = a_0 + a_1\frac{x}{w} + a_2(\frac{x}{w})^2$$

on the affine chart with coordinates $(\frac{x}{w}, \frac{w}{y} := \eta)$. Using these coordinates at infinity it will be easy to see when some regions in the plane "meet" at infinity in $\mathbb{F}_2$.

We shall now look for normal forms of Equation (2). Singular points of $X'(\mathbb{R})$ are in one-to-one correspondence with singular points of $B(\mathbb{R})$. The different cases we shall now consider are distinguished by the number of irreducible components of the trisection B.

Three components. In this case, we shall see that any connected component of the topological normalization of any real double cover ramified over B will have at most 4 singular points. Observe that at least one of the three components is real. Equation (2) becomes

$$(y - \alpha(x))(y - \beta(x))(y - \gamma(x)) = 0$$

and, changing real coordinates for $Q = \mathbb{P}(1, 1, 2)$, we may assume $\gamma = 0$. The case $\beta = \overline{\alpha}$ where two components are complex conjugate leads to at most 2 singular points: $\operatorname{Re}\alpha(x) = 0, y = \operatorname{Im}\alpha(x)$. We can therefore assume that the three components are real. Thus Equation (2) becomes $(y - \alpha)(y - \beta)y = 0$ where $\alpha(x) = \alpha_0 + \alpha_1x + \alpha_2x^2$ and $\beta(x) = \beta_0 + \beta_1x + \beta_2x^2$ are polynomials of degree 2.

- Assume no 2 parabolae are tangent. Then, since we can permute the 3 curves, we can fix the one which is the lowest at infinity (i.e., if we write the curves as $y = a_0 + a_1x + a_2x^2$, the one with the smallest value

of a_2). Changing coordinates, we get only the curve $y = 0$ and two convex parabolae, i.e. with $\alpha_2 > 0$ and $\beta_2 > 0$.

The 6 intersection points are distinct and given by

$$y = \alpha(x)\beta(x) = 0 , \qquad \alpha(x) = \beta(x) = y .$$

The curve B is real, thus if one of these points is not real, then the number of real singular points is bounded above by 4 and we are done. From now on, we suppose that the six points are real. Set

$$\begin{cases} \alpha(x) = \alpha_2(x - a_1)(x - a_2), \ a_1 < a_2 \ ; \\ \beta(x) = \beta_2(x - b_1)(x - b_2) \ . \end{cases} \tag{3}$$

Multiplying y possibly by β_2, we may assume $\beta_2 = 1$. We may reduce to the case $0 < \alpha_2 < 1$ by possibly exchanging the roles of α and β. We can further use a translation in the x axis and assume $b_1 = -b_2$, then (3) becomes:

$$\begin{cases} \alpha(x) = \alpha_2(x - a_1)(x - a_2), \ a_1 < a_2, \ 0 < \alpha_2 < 1 \ ; \\ \beta(x) = (x^2 - b^2), \ 0 < b \ . \end{cases}$$

Up to reflection $x \leftrightarrow -x$, this leads to 4 possibilities, namely (see Figure 1)

$$b < a_1 \ , \ -b < a_1 < b < a_2 \ , \ a_1 < -b < b < a_2 \ , \ -b < a_1 < a_2 < b \ .$$

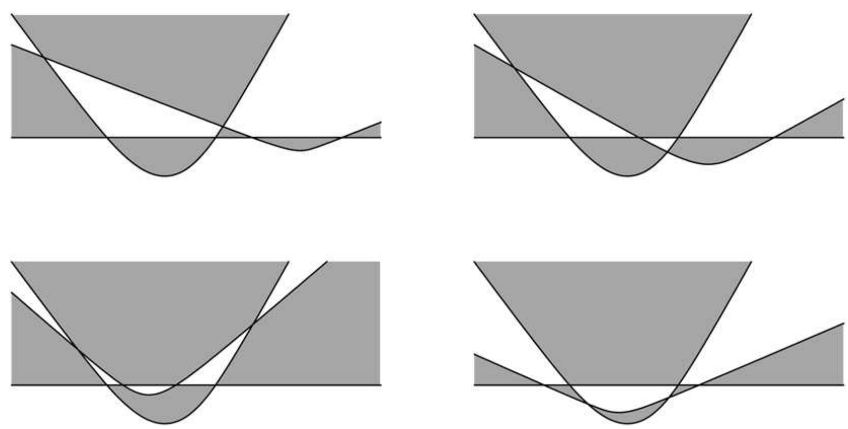

Figure 1. Three parabolae, 6 singular points.

Remark 2.2. *Observe that in these figures two components are connected at infinity if their boundaries have two unbounded arcs belonging to the same pair of parabolae.*

- Assume 2 parabolae are tangent. Then we cannot arbitrarily permute the 3 curves, and we shall have to consider furthermore the cases $\alpha_2 > 1$ and $\alpha_2 < 0$.

  Without loss of generality, the two tangent parabolae are given by $y = 0$ and $y = x^2$. The third parabola is

$$y = \alpha_2(x - a_1)(x - a_2), \ a_1, a_2 \in \mathbb{R}^*, a_1 < a_2 \ .$$

If $\alpha_2 > 0$, again using the reflection $x \leftrightarrow -x$, we are lead to only 3 possibilities which are degenerate cases of the preceding ones (by possibly exchanging the roles of α and β). In fact, if a_1, a_2 have opposite signs and $\alpha_2 \leq 1$, then the two parabolae $y = x^2$, $y = \alpha_2(x-a_1)(x-a_2)$ do not meet in real points.

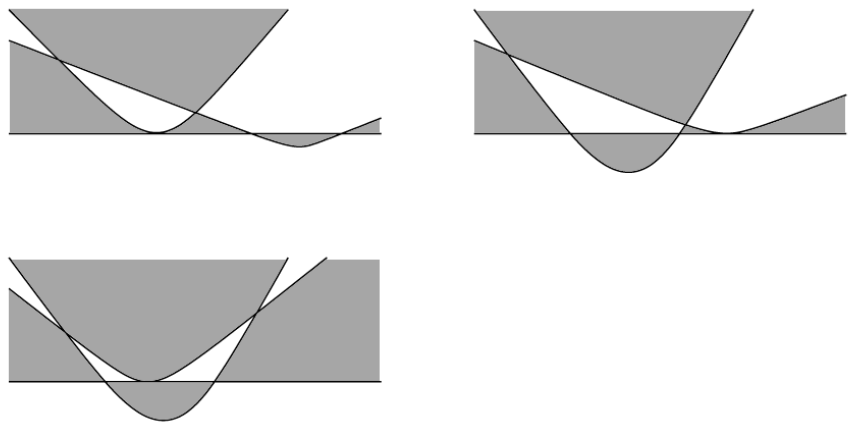

FIGURE 2. Three parabolae, 5 singular points.

If $\alpha_2 < 0$ this leads to 2 possibilities, up to reflection $x \leftrightarrow -x$.

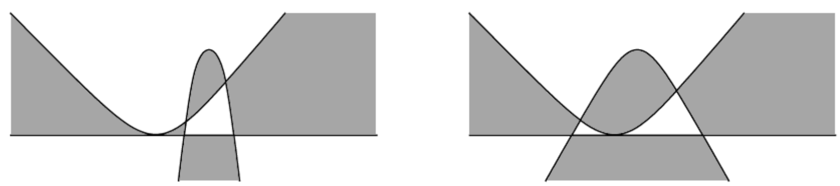

FIGURE 3. Three parabolae, 5 singular points.

Two components. Here, we will see that in most cases, any connected component of the topological normalization of any real double cover ramified over B has at most 4 singular points. There will remain two cases to examine separately, see Figures 6 and 7. Equation (2) becomes

$$(y - \alpha(x))(y^2 - \gamma(x)) = 0 .$$

If the bisection $y^2 - \gamma(x) = 0$ is smooth, then the number of singular points is bounded from above by 4. Hence we assume the bisection to have a singular point O at $x = y = 0$. To ensure that the bisection and the parabola have 4 real intersection points, the polynomial $\alpha(x)^2 - \gamma(x)$ must have 4 distinct roots. These roots are all supposed to be real and non vanishing in order that B have 5 singular points. The singular point O is either nodal or cuspidal.

If O is an ordinary double point, the bisection is given by $y^2 - x^2h(x) = 0$ where the quadratic polynomial h is not a square since the bisection is irreducible. Changing coordinates, we can assume that the parabola is given by $y = 0$ and the bisection C by $(y + \alpha(x))^2 - x^2h(x) = 0$. Without loss of generality, $\alpha(0) > 0$.

Observe that the leading coefficient of h is non vanishing since the curve C does not pass through the vertex of Q.

The number of real singularities implies that h is not always negative and $h(0) \neq 0$. If $h(0) < 0$, then O will be isolated in $B(\mathbb{R})$ and gives rise to a globally nonseparating point of the double covering X', in view of the following.

Remark 2.3. *Let $\pi\colon X'(\mathbb{R}) \to F(\mathbb{R})$ be a double cover of a smooth connected real surface $F(\mathbb{R})$. If b is an isolated point of the real branch curve $B(\mathbb{R})$, then either $p = \pi^{-1}(b)$ is an isolated point of $X'(\mathbb{R})$, or p is a locally separating point of $X'(\mathbb{R})$. If however $B(\mathbb{R})$ has a component Γ of dimension 1, then p is globally nonseparating.*

Proof. Take a path connecting b to Γ.

□

If $h(0) > 0$, and the function h is somewhere negative, observe that $y = 0$ disconnects the cylinder $\mathbb{F}_2(\mathbb{R}) - \Sigma_\infty(\mathbb{R})$. Since the polynomial $\alpha(x)^2 - x^2h(x)$ is assumed to have 4 distinct roots, up to taking a projectivity of $\mathbb{P}^1(\mathbb{R})$ sending ∞ to a finite point, we see that there is only one topological possibility, given by Figure 4.

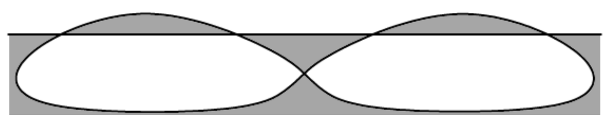

FIGURE 4. Two irreducible components.

If $h(x) > 0$, for all x, then C is a double cover of $\mathbb{P}^1(\mathbb{R})$, and we can write C as $C^u \cup C^l$, where C^u is the "upper part", C^l the lower part. Because of our choice $\alpha(0) > 0$, $C^u \cap \{y = 0\} = \emptyset \Rightarrow C \cap \{y = 0\} = \emptyset$, hence there are two cases: $\#(C^u \cap \{y = 0\}) = 2$, given by Figure 5, $\#(C^u \cap \{y = 0\}) = 4$, given by Figure 6.

After we describe the branch curve B, observe that we obtain two different surfaces multiplying the equation of B by ± 1. In Figure 5, any connected component of the topological normalization of any double cover will have at most 4 singular points. In Figure 6, for only one choice of sign, the topological normalization of the double cover will have a connected component with 5 singular points. For this double cover however the singular point O turns out to be a globally nonseparating A_1 singular point hence does not belong to $\mathcal{P}_{X'}$.

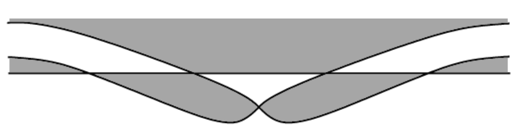

FIGURE 5. Two irreducible components.

If O is a cusp, the equation of B is

$$(y - \alpha(x))(y^2 - x^3 l(x)) = 0 .$$

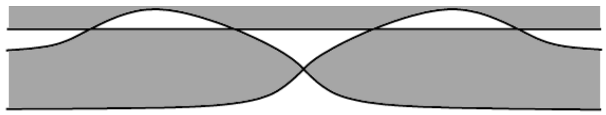

FIGURE 6. The point O is a globally nonseparating A_1 singular point.

Using a dilatation $y \mapsto \lambda y$ and possibly the usual reflection $x \leftrightarrow -x$, we may assume $l(0) = 1$ and the equation of the bisection becomes

$$y^2 - x^3(1 + ax) = 0 .$$

To ensure that the bisection and the parabola have 4 real intersection points, the equation $\alpha(x)^2 - x^3(1 + ax) = 0$ must have 4 distinct roots. Possibly changing the line $x = \infty$ via a projectivity, we may assume that $a > 0$ and indeed $a = 1$. It is easy then to see that the only possible configuration is given by Figure 7.

Recall that we obtain two different surfaces multiplying the equation of B by ± 1. For only one choice of sign the topological normalization of the double cover will have a connected component with 5 singular points. For this double cover, however, the point O turns out to be of real type A_2^- which does not belong to $\mathcal{P}_{X'}$.

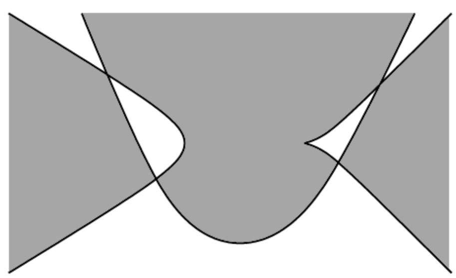

FIGURE 7. The cusp gives rise to a singular point of type A_2^-.

One component. If the trisection is irreducible, then it has at most 4 singular points, since $B(\mathbb{C})$ has genus 4.

Proof of Proposition 2.1. We proceed according to the number of irreducible components of B, recalling that the singular points of X correspond to the singular points of B.

If B is irreducible, we have already seen that B has at most 4 singular points.

If instead B has 2 irreducible components, and B has strictly more than 4 singular points, we have seen that B has exactly 5 singular points, and that the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ has one of the topological configurations of Figures 4, 5, 6, 7.

In the case of Figure 4 none of the connected components of the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ contains more than 4 points.

The same occurs for the case of Figure 5, while for Figure 6 there is exactly one connected component D containing the 5 singular points. However, in this

case the nodal point of the bisection yields a globally nonseparating singular point of X' for the choice of the positivity region which includes D.

Similarly, for the case of Figure 7 there is exactly one connected component D containing the 5 singular points. However, in this case the cuspidal point yields a point of type A_2^- which does not belong to $\mathcal{P}_{X'}$.

Assume now that B has 3 irreducible components, and at least 5 singular points.

If there are 6 singular points, the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ has one of the topological configurations of Figure 1, and none of the connected components of the complement $\mathbb{F}_2(\mathbb{R}) \setminus B(\mathbb{R})$ contains more than 4 points.

An easy inspection of Figures 2 and 2 reveals that the same holds also in the remaining cases.

□

Proposition 2.4 (Kollár)**.** *Let X be a real conic bundle with X Du Val.*

Then $m_i \leq 4$, $i = 1, 2, \ldots, r$. Moreover, if $m_i = 4$, then $\bar{n}(M_i) \cap \mathcal{P}_X$ contains 4 A_1 points. Whereas, if $m_i = 3$, then $\bar{n}(M_i) \cap \mathcal{P}_X$ contains at least 2 A_1.

Proof. The assertion $m_i \leq 4$ is the last assertion of the proof of cor. 9.8 of [Kol99b]. But the same argument proves indeed what we have stated above. □

Proof of Theorem 0.1. Recall that by 1.7 the numbers $m_1, \ldots m_r$ of Du Val singular points on the connected components of $\overline{X(\mathbb{R})}$ which are not of type A^- and globally nonseparating is an invariant by extremal birational contractions. Hence, by Theorem 1.2 it suffices to consider the case where X is either a conic bundle or a Del Pezzo surface.

The case of a conic bundle is settled by Proposition 2.4, and by virtue of Lemma 1.8 it suffices to consider the case where X is a Du Val Del Pezzo surface of degree 1.

Now it suffices to apply Proposition 2.1.

□

3. Real rationally connected Threefolds

This section is devoted to the proof of Corollary 0.2. We first of all introduce the concept of a Werther fibration (cf. [HM05b]), which allows us to set the integer k on an equal footing in both cases 1) and 2).

Let $S^1 \times D^2$ be the *solid torus*, where S^1 is the unit circle $\{u \in \mathbb{C} \mid |u| = 1\}$ and D^2 is the closed unit disc $\{z \in \mathbb{C}, |z| \leq 1\}$. A *Seifert fibration* of the solid torus is a differentiable map of the form

$$f_{p,q} : S^1 \times D^2 \to D^2 \ , (u, z) \mapsto u^q z^p \ ,$$

where p and q are natural integers, with $p \neq 0$ and $\gcd(p, q) = 1$. Let N be a 3-manifold. A *Seifert fibration* of N is a differentiable map f from N into a differentiable surface S having the following property. Every point $P \in S$ has a

closed neighborhood U such that the restriction of f to $f^{-1}(U)$ is diffeomorphic to a Seifert fibration of the solid torus.

Let A^2 be the half-open annulus $\{w \in \mathbb{C} \mid 1 \leq |w| < 2\}$. Let P be the differentiable 3-manifold defined by $P = \{((w,z) \in A^2 \times \mathbb{C} \mid |z|^2 = |w|^2 - 1\}$. Let $\omega\colon P \to A^2$ be the projection defined by $\omega(w,z) = w$. It is clear that ω is a differentiable map, that ω is a trivial circle bundle over the interior of A^2, and that ω is a diffeomorphism over the boundary of A^2.

Definition 3.1. *Let $g\colon N \to F$ be a differentiable map from a 3-manifold N without boundary into a differentiable surface F with boundary. The map g is a* Werther fibration *if*

1. *the restriction of g over the interior of F is a Seifert fibration, and*
2. *every point x in the boundary of F has an open neigborhood U such that the restriction of g to $g^{-1}(U)$ is diffeomorphic to the restriction of ω over an open neighborhood of 1 in A^2.*

This definition was introduced in [HM05b], and is motivated by the following theorem.

Theorem 3.2. [HM05b, Theorem 2.6] *Let N be a 3-dimensional compact manifold without boundary. Then N is a Seifert fibred manifold or a connected sum of finitely many lens spaces if and only if there is a Werther fibration $g\colon N \to F$ over a compact connected differentiable surface F with boundary. Furthermore N is Seifert fibred if and only if there exist such a map $g\colon N \to F$ with $\partial F = \emptyset$.*

Thanks to the Minimal Model Program over $\mathbb{R}$ ([Ko99a]), the original setting for $f\colon W \to X$ in Corollary 0.2 is replaced by the following: W is a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$, $W(\mathbb{R})$ is a topological 3-manifold, and $f\colon W \to X$ is a rational curve fibration over $\mathbb{R}$ such that $-K_W$ is f-ample.

The following result relates the connected components of $W(\mathbb{R})$ with the connected components of the topological normalization $\overline{X(\mathbb{R})}$.

Proposition 3.3. [Kol99b, Cor. 6.8] *Let W be a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$. Let $f\colon W \to X$ be a rational curve fibration over $\mathbb{R}$ such that $-K_W$ is f-ample. Let $N \subset W(\mathbb{R})$ be a connected component. Then $f(N)$ intersects only one of the connected components of $X(\mathbb{R}) \setminus \operatorname{Sing} X$.*

Let $\overline{n}\colon \overline{W(\mathbb{R})} \to W(\mathbb{R})$ be the topological normalization. The following is the key result which relates the integer $k(N)$ which was defined above to the numbers m_i of the singularities in $\mathcal{P}_X \cap M_i$.

Proposition 3.4. [Kol99b, Th. 8.1(6)] *Let W be a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$. Let $f\colon W \to X$ be a rational curve fibration over $\mathbb{R}$ such that $-K_W$ is f-ample. Let N be a connected component of the topological normalization $\overline{W(\mathbb{R})}$ and assume that N is an orientable topological 3-manifold. Then there exists a small perturbation $g\colon N \to F$ of $f|_{\overline{n}(N)}$ which is a Werther fibration. Furthermore, there is an*

injection from the set of multiple fibres of g to the set of singular points of X contained in $f(\overline{n}(N))$ which are of real type A^+_μ, $\mu \geq 1$. If $\partial F = \emptyset$, then g is a Seifert fibration. If $\partial F \neq \emptyset$, then N is a connected sum of lens spaces and the number of lens spaces is equal to the number of multiple fibres of g.

Proof of Corollary 0.2. We want to show that, for each component N of $\overline{W(\mathbb{R})}$, we have $k(N) \leq 4$. From the above Proposition 3.4 it follows that $k(N)$ is the number of multiple fibres of the Werther fibration, hence it suffices to bound the number of singular points of X contained in $f(\overline{n}(N))$ which are of real type A^+_μ. If $f(\overline{n}(N))$ is not a connected component of $X(\mathbb{R}) \setminus \operatorname{Sing} X$, then from [Kol99b, 8.2], N is a connected sum of lens spaces and $f(\overline{n}(N))$ may contain some globally nonseparating singular points of type $A^+_1 \sim A^-_1$. These produce double fibres for g, which however correspond to lens space summands $\mathbb{P}^3(\mathbb{R})$. These summands are excluded by the maximality of a in the definition of $k(N)$. Thus, by Proposition 3.3, it suffices to bound the number of singular points of X contained in $f(\overline{n}(N))$ which are of real type A^+_μ and globally separating. Since however these points are a subset of $\mathcal{P}_X \cap M_i$, for some $i \in \{1, \dots r\}$, the desired inequality follows from Theorem 0.1.

□

4. Seifert fibrations over a torus

This section is devoted to the proof of Theorem 0.3.

Lemma 4.1. *Let $r \colon X \to \mathbb{P}^1_{\mathbb{R}}$ be a real conic bundle. Suppose that X is a Du Val surface. If $\overline{X(\mathbb{R})}$ has a connected component M diffeomorphic to $S^1 \times S^1$, then r is smooth along $X(\mathbb{R})$ and $X(\mathbb{R}) \sim \overline{X(\mathbb{R})} \sim S^1 \times S^1$.*

Proof. We first want to show that $r_M := r \circ \overline{n}|_M$ is surjective and that $\overline{X(\mathbb{R})} \sim S^1 \times S^1$. Assume that r_M is not surjective. Then $\operatorname{Im}(r_M)$ is homeomorphic to a segment $[a, b]$. The fibres $r_M^{-1}(a)$ and $r_M^{-1}(b)$ are the ends of $r_M^{-1}(a, b)$ and they have a (punctured) tubular neighbourhood which is homeomorphic to an annulus. This shows that $r_M^{-1}(a)$, $r_M^{-1}(b)$ are connected. The fibre $r_M^{-1}(a)$ is a simplicial complex of dimension ≤ 1, and if $S^1 \subset r_M^{-1}(a)$, then S^1 has a (punctured) tubular neighbourhood which is connected, contradicting the orientability of M. Hence $r_M^{-1}(a)$, $r_M^{-1}(b)$ have Euler number 1.

It suffices to show that each fibre $r_M^{-1}(t)$, $t \in (a, b)$ has Euler number ≥ 0 and we obtain a contradiction to $e(M) = 0$. Looking at the normal forms for singular points of type A_μ for conic bundles (given in [Kol99b, Proof of Cor. 9.8]), we see that every fibre of $\overline{r} := r \circ \overline{n}$ is either a circle (and then r is smooth on the fibre), or a point, or an interval. Thus r_M is surjective.

Again, the Euler number argument shows that all fibres of $\overline{r}$ are circles, hence r is smooth on $X(\mathbb{R})$ and $M \sim \overline{X(\mathbb{R})} \sim X(\mathbb{R})$. □

Proposition 4.2. *Let X be a real Du Val surface which is rational over $\mathbb{C}$. Assume that $X(\mathbb{R})$ contains only singularities of type A^+_μ and that $\overline{X(\mathbb{R})}$ contains a connected component diffeomorphic to $S^1 \times S^1$. Then $\overline{X(\mathbb{R})}$ is connected, thus $\overline{X(\mathbb{R})} \sim S^1 \times S^1$. Furthermore, there is a minimal model of X which is a real conic bundle over $\mathbb{P}^1$.*

Proof. Let X be as above. The blow-up of a point of type A^+_μ for $\mu \geq 2$ induces a homeomorphism between the real parts. Thus there is a surface Z such that all singular points are of type A_1 and $\overline{Z(\mathbb{R})}$ is homeomorphic to $\overline{X(\mathbb{R})}$. Let Z^* be a Du Val minimal model of Z. Then by [Kol99b, Th. 9.6], $\pi\colon Z \to Z^*$ is the composition of inverses of weighted blow-ups (of smooth points). Hence π is an isomorphism with the exception of a finite number of smooth points $p_1, \dots, p_s \in Z^*$ at which one takes the weighted blow-up which, in local coordinates (x, y) around p_j, has the form $\{xu - vy^2\} \to \{(x, y)\}$. Since the weighted blow-up produces globally nonseparating points, there is a bijection between the connected components of $\overline{Z(\mathbb{R})}$ and the connected components $\overline{Z^*(\mathbb{R})}$. The weighted blow-up followed by topological normalization on a disc neighbourhood of p_j has the effect of replacing p_j by a closed segment. Hence the connected component of $\overline{Z^*(\mathbb{R})}$ coming from the one of $\overline{Z(\mathbb{R})}$ diffeomorphic to $S^1 \times S^1$ is again diffeomorphic to $S^1 \times S^1$. Observe again that the singularities of Z^* are only of type A_1.

We have two cases:

(1) The minimal model Z^* is a real Del Pezzo surface of degree 1 or 2;
(2) Z^* is a real conic bundle.

In case (1), we have a realization of Z^* as a double cover, and the topological normalization $\overline{Z^*(\mathbb{R})}$ can be realized as the real part of a real perturbation Z^*_ε of Z^* (by Brusotti's Theorem, [Bru21]). The surface Z^*_ε is a smooth real Del Pezzo surface of degree 1 or 2. An orientable connected component of such a surface is a sphere, see e.g. [Sil89, Chap. 3], so this case does not occur.

Case (2) follows from Lemma 4.1. □

To prove Theorem 0.3, we need the conclusion of Proposition 4.2 in a more general setting. First, we give a partial generalisation of Brusotti's theorem in the case of a Du Val Del Pezzo surface.

Theorem 4.3. *Let X be a Du Val Del Pezzo surface. One can obtain, by a global small deformation of X, all the possible smoothings of the singular points of X.*

Proof. The main theorem on deformations of compact complex spaces was proven in [Gra74]. Good references are [Pa76] and [Pi81]. The tangent space to $\mathrm{Def}(X)$ is given by $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$, see [Se06, Cor. 1.1.11]. The obstruction space $\mathrm{Ob}(X)$ is given by $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$, see [Se06, Prop. 2.4.8].

By the local to global spectral sequence for Ext, we have the following exact sequence

$$H^1(X, \Theta_X) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)) \to H^2(X, \Theta_X) \to 0$$

and $\mathrm{Ob}(X) = H^2(X, \Theta_X)$.

Therefore the vanishing $H^2(X, \Theta_X) = 0$ implies:

- The local deformation space is smooth
- Global deformations map onto local deformations.

We use a calculation by Burns and Wahl [BW74, Prop. 1.2], to the effect that, if S is the minimal resolution of the Du Val singularities of X, then $p_*(\Theta_S) = \Theta_X$. Whence, $H^2(X, \Theta_X) = H^2(S, \Theta_S)$.

But the dual space of $H^2(S, \Theta_S)$ is $H^0(S, \Omega^1_S(K_S))$. The conclusion follows from $H^0(S, \Omega^1_S(K_S)) = 0$ since $H^0(S, \Omega^1_S) = 0$ and $H^0(S, \mathcal{O}_S(-K_S)) \neq 0$.

□

Lemma 4.4. *Consider a real singular point of a surface X, of local equation $z^2 = f(x, y)$ where f vanishes at the origin and has there an isolated singular point which we assume to be a nonisolated real point. Then the topological normalization of $X(\mathbb{R})$ is locally homeomorphic to the real part $X_\varepsilon(\mathbb{R})$ of the surface X_ε with equation $z^2 = f(x, y) - \varepsilon$, for ε sufficiently small and positive.*

Proof. The real curve $f(x, y) = 0$ has $2m$ arcs entering into the singular point, ordered counterclockwise, and the region of positivity consists of m sectors, which alternate themselves to the m sectors of negativity. Furthermore, we have $m \neq 0$ because the origin is a nonisolated real point of the curve. The smooth curve $f(x, y) = \varepsilon$ determines m domains of positivity whose closure is homeomorphic to the closure of the corresponding sector of positivity of $f(x, y)$ (where it is contained). It follows right away that the double cover $z^2 = f(x, y) - \varepsilon$ replaces the singular point by m points, one for each connected component of $X(\mathbb{R}) \setminus \{0\}$. □

Proposition 4.5. *Let X be a real Du Val surface which is rational over $\mathbb{C}$. Assume that all locally separating singularities are gobally separating and that $\overline{X(\mathbb{R})}$ contains a connected component diffeomorphic to $S^1 \times S^1$. Then $\overline{X(\mathbb{R})}$ is connected, thus $\overline{X(\mathbb{R})} \sim S^1 \times S^1$. Furthermore, there is a minimal model of X which is a real conic bundle over $\mathbb{P}^1$.*

Proof. The minimal resolution of a singular point of type A^-_μ, μ even, induces a homeomorphism between the real parts, thus as in the proof of 4.2, there is a surface Z such that

- all singular points are of type A_1, or of type $A^-_\mu, \mu > 1$, μ odd, or not of type A, and
- $\overline{Z(\mathbb{R})}$ is homeomorphic to $\overline{X(\mathbb{R})}$.

Let Z^* be a Du Val minimal model of Z. Suppose that the minimal model Z^* is a real Del Pezzo surface of degree 1 or 2. We have then a realization of Z^* as a double cover and we can apply Remark 2.3 to exclude singular points which are isolated real points of the branch curve. By Lemma 4.4 and Theorem 4.3, the topological normalization $\overline{Z^*(\mathbb{R})}$ can be realized as the real part of a real perturbation Z^*_ε of Z^*. The surface Z^*_ε is a smooth real Del Pezzo surface of degree 1 or 2. An orientable connected component of such a surface is a sphere, so this case does not occur.

Hence Z^* is a real conic bundle and the conclusion follows from Lemma 4.1.

□

Proof of Theorem 0.3. The component N of $W(\mathbb{R})$ is Seifert fibred hence $f(N)$ is the closure of a connected component of $X(\mathbb{R}) \setminus \operatorname{Sing} X$ (see the statement in [Kol99b, 8.2] "so we are in the case (4)"). In the proof of [loc. cit. 6.8], using

[loc. cit. 4.3], Kollar claims that $f(N)$ cannot map to both local components of a locally separating singularity. Whence, this singularity must be globally separating. Thus all singularities of $f(N)$ which are locally separating are gobally separating. We are now in the situation of Proposition 4.5 whence $\overline{X(\mathbb{R})} \sim S^1 \times S^1$. Furthermore, the minimal model Z^* is a real conic bundle and Lemma 4.1 gives that Z^* is smooth along $Z^*(\mathbb{R})$, thus $Z(\mathbb{R}) \cap \mathcal{P}_Z = \emptyset$, hence $X(\mathbb{R}) \cap \mathcal{P}_X = \emptyset$.

Applying Proposition 3.4 on a minimal model of $W \to X$, we get the conclusion. □

References

[Bru21] L. Brusotti, *Sulla "piccola variazione" di una curva piana algebrica reale*, **Rend. Rom. Acc. Lincei 30** (1921) 375–379

[BW74] D. M. Burns, J. Wahl, *Local contributions to global deformations of surfaces*, **Inventiones Math. 26** (1974) 67–88

[Co14] A. Comessatti, *Sulla connessione delle superfizie razionali reali*, **Annali di Math. 23** (1914) 215–283

[Dem80] M. Demazure, *Surfaces de Del Pezzo, II, III, IV et V*, In: Séminaire sur les singularités des surface (Demazure, Pinkham, Teissier eds.) **Lecture Notes in Math. 777**, Springer-Verlag 1980

[DIK00] A. Degtyarev, I. Itenberg, V. Kharlamov, *Real Enriques Surfaces*, **Lecture Notes in Math. 1746**, Springer-Verlag 2000

[Gra74] H. Grauert, *Der Satz von Kuranishi für kompakte komplexe Räume*, **Invent. Math. 25** (1974) 107-142

[Hem76] J. Hempel, *3-Manifolds*, **Ann. Math. Studies 86** Princeton Univ. Press 1976

[HM05] J. Huisman, F. Mangolte, *Every orientable Seifert 3-manifold is a real component of a uniruled algebraic variety*, **Topology 44** (2005) 63–71

[HM05b] J. Huisman, F. Mangolte, *Every connected sum of lens spaces is a real component of a uniruled algebraic variety*, **Ann. Fourier 55** (2005) 2475–2487

[Ko98] J. Kollár, *Real algebraic threefolds. I. Terminal singularities Dedicated to the memory of Fernando Serrano*, **Collect. Math. 49** (1998), no. 2-3, 335–360.

[Ko99a] J. Kollár, *Real algebraic threefolds. II. Minimal model program*, **J. Amer. Math. Soc. 12** (1999) 33–83

[Kol99b] J. Kollár, *Real algebraic threefolds. III. Conic bundles*, **J. Math. Sci. (New York) 94** (1999) 996–1020

[Ko00] J. Kollár, *Real algebraic threefolds. IV. Del Pezzo fibrations*, **Complex analysis and algebraic geometry**, 317–346, de Gruyter, Berlin, 2000.

[Ko01] J. Kollár, *The topology of real algebraic varieties*, **Current developments in mathematics, 2000**, 197–231, Int. Press, Somerville, MA, 2001.

[Mil62] J. Milnor, *A unique factorization theorem for 3-manifolds*, **Amer. J. Math. 84** (1962) 1–7

[Pa76] V. P. Palamodov, *Deformations of complex spaces (Russian)*, **Uspehi Mat. Nauk. 31** (1976), no. 3 (189), 129–194.

[Pi81] H. Pinkham, *Some local obstructions to deforming global surfaces*, Leopoldina Symposium: Singularities (Thringen, 1978) **Nova Acta Leopoldina (N.F.) 52** (1981) no. 240, 173–178.

[Se06] E. Sernesi, *Deformations of algebraic schemes*, **Grundlehren der Mathematischen Wissenschaften, 334**, Springer-Verlag 2006

[Sil89] R. Silhol, *Real Algebraic Surfaces*, **Lecture Notes in Math. 1392** Springer-Verlag 1989

Author's address:

Fabrizio Catanese
Lehrstuhl Mathematatik VIII
Universität Bayreuth, NWII
D-95440 Bayreuth, Germany

e-mail: Fabrizio.Catanese@uni-bayreuth.de

Frédéric Mangolte
Laboratoire de Mathématiques
Université de Savoie
73376 Le Bourget du Lac Cedex, France

e-mail: Frederic.Mangolte@univ-savoie.fr